Paper Reference(s)

6669/01

Edexcel GCE

Further Pure Mathematics FP3

Advanced/Advanced Subsidiary

Monday 23 June 2014 – Morning

Time: 1 hour 30 minutes

Materials required for examination	**Items included with question papers**
Mathematical Formulae (Pink)	Nil

Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation or symbolic differentiation/integration, or have retrievable mathematical formulae stored in them.

Instructions to Candidates

In the boxes above, write your centre number, candidate number, your surname, initials and signature.
Check that you have the correct question paper.
Answer ALL the questions.
You must write your answer for each question in the space following the question.
When a calculator is used, the answer should be given to an appropriate degree of accuracy.

Information for Candidates

A booklet 'Mathematical Formulae and Statistical Tables' is provided.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 9 questions in this question paper. The total mark for this paper is 75.
There are 32 pages in this question paper. Any blank pages are indicated.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.

P43147A

1. The line l passes through the point $P(2, 1, 3)$ and is perpendicular to the plane Π whose vector equation is

$$\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 3$$

Find

(*a*) a vector equation of the line l,

(2)

(*b*) the position vector of the point where l meets Π.

(4)

(*c*) Hence find the perpendicular distance of P from Π.

(2)

2.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}$$

(*a*) Show that matrix **M** is not orthogonal.

(2)

(*b*) Using algebra, show that 1 is an eigenvalue of **M** and find the other two eigenvalues of **M**.

(5)

(*c*) Find an eigenvector of **M** which corresponds to the eigenvalue 1.

(2)

The transformation $M : {\sim}^{3} \to {\sim}^{3}$ is represented by the matrix **M**.

(*d*) Find a cartesian equation of the image, under this transformation, of the line

$$x = \frac{y}{2} = \frac{z}{-1}$$

(4)

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3. Using calculus, find the exact value of

(*a*) $\int_1^2 \frac{1}{\sqrt{(x^2 - 2x + 3)}}\,\mathrm{d}x$

(4)

(*b*) $\int_0^1 \mathrm{e}^{2x} \sinh x \,\mathrm{d}x$

(4)

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4. Using the definitions of hyperbolic functions in terms of exponentials,

(*a*) show that

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

(3)

(*b*) solve the equation

$$4\sinh x - 3\cosh x = 3$$

(4)

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5. Given that $y = \operatorname{artanh} \frac{x}{\sqrt{(1 + x^2)}}$

show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{(1 + x^2)}}$

(4)

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6. [*In this question you may use the appropriate trigonometric identities on page 6 of the pink Mathematical Formulae and Statistical Tables*.]

The points $P(3\cos\alpha, 2\sin\alpha)$ and $Q(3\cos\beta, 2\sin\beta)$, where $\alpha \neq \beta$, lie on the ellipse with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

(*a*) Show the equation of the chord PQ is

$$\frac{x}{3}\cos\frac{(\alpha+\beta)}{2} + \frac{y}{2}\sin\frac{(\alpha+\beta)}{2} = \cos\frac{(\alpha-\beta)}{2}$$

(4)

(*b*) Write down the coordinates of the mid-point of PQ.

(1)

Given that the gradient, m, of the chord PQ is a constant,

(*c*) show that the centre of the chord lies on a line

$$y = -kx$$

expressing k in terms of m.

(5)

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7. A circle C with centre O and radius r has cartesian equation $x^2 + y^2 = r^2$ where r is a constant.

(*a*) Show that $1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{r^2}{r^2 - x^2}$.

(3)

(*b*) Show that the surface area of the sphere generated by rotating C through π radians about the x-axis is $4\pi r^2$.

(5)

(*c*) Write down the length of the arc of the curve $y = \sqrt{(1 - x^2)}$ from $x = 0$ to $x = 1$.

(1)

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8. The position vectors of the points A, B and C from a fixed origin O are

$$\mathbf{a} = \mathbf{i} - \mathbf{j}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{c} = 2\mathbf{j} + \mathbf{k}$$

respectively.

(*a*) Using vector products, find the area of the triangle ABC.

(4)

(*b*) Show that $\frac{1}{6}\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0$.

(3)

(*c*) Hence or otherwise, state what can be deduced about the vectors **a**, **b** and **c**.

(1)

9.

$$I_n = \int \left(x^2 + 1\right)^{-n} \mathrm{d}x, \quad n > 0$$

(*a*) Show that, for $n > 0$,

$$I_{n+1} = \frac{x\left(x^2 + 1\right)^{-n}}{2n} + \frac{2n - 1}{2n} I_n$$

(5)

(*b*) Find I_2.

(3)

TOTAL FOR PAPER: 75 MARKS

END

Question	Scheme		Marks
	$P(2, 1, 3)$ and $\mathbf{r}.(\mathbf{i}-2\mathbf{j}-\mathbf{k})=3$		
1.(a)	l is parallel to $\mathbf{i}-2\mathbf{j}-\mathbf{k}$	An appreciation that $\mathbf{i}-2\mathbf{j}-\mathbf{k}$ is the direction of the line (may be implied).	M1
	$"\mathbf{r}"=2\mathbf{i}+\mathbf{j}+3\mathbf{k}+t(\mathbf{i}-2\mathbf{j}-\mathbf{k})$ or $(\mathbf{r}-(2\mathbf{i}+\mathbf{j}+3\mathbf{k}))\times(\mathbf{i}-2\mathbf{j}-\mathbf{k})=\mathbf{0}$ or $\mathbf{r}\times(\mathbf{i}-2\mathbf{j}-\mathbf{k})=(2\mathbf{i}+\mathbf{j}+3\mathbf{k})\times(\mathbf{i}-2\mathbf{j}-\mathbf{k})$ or $\mathbf{r}\times(\mathbf{i}-2\mathbf{j}-\mathbf{k})=5\mathbf{i}+5\mathbf{j}-5\mathbf{k}$	A correct vector **<u>equation</u>** in any form. (Allow any multiple of the direction vector.)	A1
			(2)
(b)	$\begin{pmatrix}2+t\\1-2t\\3-t\end{pmatrix}\begin{pmatrix}1\\-2\\-1\end{pmatrix}=3$	Substitutes a parametric form of their line from part (a) into the equation of the plane. This statement is sufficient.	M1
	$2+t-2(1-2t)-(3-t)=3$	Correct equation **(allow unsimplified)**	A1
	$2+t-2+4t-3+t=3\Rightarrow t=......$	Solves to find a value for t **Dependent on the first M**	**d**M1
	$t=1\Rightarrow l$ meets Π at $\mathbf{r}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}$	Correct position vector (allow as coordinates (3, -1, 2))	A1
			(4)
(c) Way 1	$PQ=\lvert 3\mathbf{i}-\mathbf{j}+2\mathbf{k}-(2\mathbf{i}+\mathbf{j}+3\mathbf{k})\rvert$		
	$=\lvert\mathbf{i}-2\mathbf{j}-\mathbf{k}\rvert=\sqrt{1^2+2^2+1^2}$	Attempts the vector PQ or QP and correct Pythagoras.	M1
	$=\sqrt{6}$	Allow awrt 2.45 or $\frac{6}{\sqrt{6}}$	A1
			(2)
(c) Way 2	$t="1"\Rightarrow\overrightarrow{PQ}="1"\times(\mathbf{i}-2\mathbf{j}-\mathbf{k})$		
	$=\lvert\mathbf{i}-2\mathbf{j}-\mathbf{k}\rvert=\sqrt{1^2+2^2+1^2}$	Attempts the vector PQ using their value for t and their normal vector and correct Pythagoras.	M1
	$=\sqrt{6}$	Allow awrt 2.45 or $\frac{6}{\sqrt{6}}$	A1
			(2)
			Total 8

Question	Scheme		Marks
2.(a)	$\mathbf{MM}^T = \begin{pmatrix}1&0&2\\0&4&1\\0&5&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&4&5\\2&1&0\end{pmatrix}$	Attempts $\mathbf{MM}^T$ or $\mathbf{M}^T\mathbf{M}$ or scalar product of at least one pair of columns or attempts magnitude of at least one column or finds det**M** or attempts $\mathbf{M}^{-1}$	M1
	$= \begin{pmatrix}5&2&0\\2&17&20\\0&20&25\end{pmatrix} \ne \mathbf{I}$ $\therefore$ **M** not orthogonal.	or scalar product $\ne 0$ or magnitude $\ne 1$ or det**M** $\ne \pm 1$ (**must see** ±) or $\mathbf{M}^{-1} \ne \mathbf{M}^T$ **<u>and</u>** conclusion. Note that not all of $\mathbf{MM}^T$ or $\mathbf{M}^{-1}$ is necessary and there may be errors but there must be some correct work (at least one correct relevant element). NB det**M** = -5. See extra notes for $\mathbf{M}^{-1}$	A1
			(2)
(b)	$\begin{vmatrix}1-\lambda&0&2\\0&4-\lambda&1\\0&5&-\lambda\end{vmatrix} = 0$	This statement is sufficient. (allow other brackets provided the determinant is implied later)	M1
	$(1-\lambda)\left[(4-\lambda)(-\lambda)-5\right]\left(-0(0-0)+2(0-0)\right)=0$		M1
	Attempts characteristic equation (= 0 may be implied by their value(s) for λ) Allow one slip e.g. – usually the omission of the "-5"		
	$(1-\lambda)((4-\lambda)(-\lambda)-5)=0$		
	$\lambda = 1$	$\lambda = 1$ with **no errors**	A1**cso**
	$\lambda^2 - 4\lambda - 5 = 0$ $\Rightarrow \lambda = 5, \lambda = -1$	M1: Attempts to find the other 2 eigenvalues from their characteristic equation by solving a 3 term quadratic. A1: $\lambda = 5, \lambda = -1$	M1A1
			(5)
(c)	$\begin{pmatrix}1&0&2\\0&4&1\\0&5&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ or $\begin{pmatrix}0&0&2\\0&3&1\\0&5&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$	A correct statement for the eigenvalue 1. (May be implied by correct equations)	M1
	$\alpha(\mathbf{i}+0\mathbf{j}+0\mathbf{k})$ where α is a constant	Any vector of this form.	A1
			(2)
(d)	$\begin{pmatrix}1&0&2\\0&4&1\\0&5&0\end{pmatrix}\begin{pmatrix}t\\2t\\-t\end{pmatrix} = \begin{pmatrix}-t\\7t\\10t\end{pmatrix}$	M1: Attempt to multiply the parametric form or direction of the line by M. **Condone use of 0 for the *x* component but the line <u>must pass through the origin.</u>** A1: Correct image vector with or without "*t*"	M1A1
	Cartesian equation $\frac{x}{-1} = \frac{y}{7} = \frac{z}{10}$	M1: Correct method to convert to cartesian form of a straight line **<u>passing through the origin</u>**. A1: Correct equations (any multiple)	M1A1
			(4)
			Total 13

Question	Scheme		Marks
3.(a)	$x^2-2x+3=(x-1)^2+2$	M1: Attempt to complete the square. Allow $(x-1)^2+k, k\neq 0$ A1: Correct expression	M1A1
	$\int \frac{1}{\sqrt{(x-1)^2+2}}\,dx=\alpha\,\text{arsinh}\left(f(x)\right)$	Allow $\alpha\ln\left(f(x)+\sqrt{\left(f(x)\right)^2+\beta}\right)$ (β>0)	M1
	$\left[\text{arsinh}\left(\frac{x-1}{\sqrt{2}}\right)\right]_1^2=\text{arsinh}\frac{1}{\sqrt{2}}$	Any equivalent exact form. Allow $\ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$ but no other terms e.g. arsinh(0)	A1
			(4)
(b)	$e^{2x}\sinh x=e^{2x}\left(\frac{e^x-e^{-x}}{2}\right)$	Substitutes the **correct** exponential of sinh*x*	M1
	$\frac{1}{2}(e^{3x}-e^x)$	Correct expression with powers of e combined.	A1
	$\int_0^1\frac{1}{2}(e^{3x}-e^x)\,dx=\left[\frac{1}{2}\left(\frac{1}{3}e^{3x}-e^x\right)\right]_0^1$ $=\frac{1}{2}\left(\frac{1}{3}e^3-e^1\right)-\frac{1}{2}\left(\frac{1}{3}e^0-e^0\right)$	$\int e^{px}\,dx=qe^{px}$ at least once and some correct use of the limits 0 and 1 and subtracts the right way round.	M1
	$=\left(\frac{e^3}{6}-\frac{e}{2}+\frac{1}{3}\right)$	Any exact equivalent (allow e^1) but all like terms collected but isw following a correct answer.	A1
			(4)
			Total 8
	(b) Integration by parts way 1		
	$I=\left[\frac{1}{2}e^{2x}\sinh x\right]_0^1-\int_0^1\frac{1}{2}e^{2x}\cosh x\,dx=\left[\frac{1}{2}e^{2x}\sinh x\right]_0^1-\left[\frac{1}{4}e^{2x}\cosh x\right]_0^1+\frac{1}{4}I$		
	$\frac{3}{4}I=\left[\frac{1}{2}e^{2x}\sinh x\right]_0^1-\left[\frac{1}{4}e^{2x}\cosh x\right]_0^1$		
	M1: Parts twice in the correct direction A1: A correct expression for *I* or any constant multiple of *I*		
	$\int_0^1 e^{2x}\sinh x\,dx\ =\frac{4}{3}\left(\frac{1}{2}e^2\sinh 1-\frac{1}{4}e^2\cosh 1+\frac{1}{4}\right)$ M1A1 oe		
	M1: Correct use of limits having integrated by parts twice A1: Correct expression (oe)		
	(b) Integration by parts way 2		
	$I=\left[e^{2x}\cosh x\right]_0^1-\int_0^1 2e^{2x}\cosh x\,dx=\left[e^{2x}\cosh x\right]_0^1-\left[2e^{2x}\sinh x\right]_0^1+4I$		
	$-3I=\left[e^{2x}\cosh x\right]_0^1-\left[2e^{2x}\sinh x\right]_0^1$		
	M1: Parts twice in the correct direction A1: A correct expression for *I* or any constant multiple of *I* $\int_0^1 e^{2x}\sinh x\,dx\ =-\frac{1}{3}\left(e^2\cosh 1-1-2e^2\sinh 1\right)$		
	M1: Correct use of limits having integrated by parts twice A1: Correct expression (oe)		

Question	Scheme		Marks
4.(a)	$\tanh x = \dfrac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}}$ or $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ or $\dfrac{e^{2x} - 1}{e^{2x} + 1}$		M1
	Use of the correct exponential form of tanh*x*		
	$1 - \tanh^2 x = 1 - \left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = \dfrac{(e^{2x} + e^{-2x} + 2) - (e^{2x} + e^{-2x} - 2)}{(e^x + e^{-x})^2}$		**d**M1
	Attempts $1 - \tanh^2 x$ with their tanh*x*, obtains a common denominator and expands the numerator correctly – three terms from $(a+b)^2$ at least once		
	$= \dfrac{2e^x.2e^{-x}}{(e^x + e^{-x})^2}$		
	$= \dfrac{4}{(e^x + e^{-x})^2} = \text{sech}^2 x*$	Correct completion with no errors	A1*
	Allow candidates to process both sides and 'meet in the middle' Note that it is possible to start from sech^2x and obtain 1 – tanh^2x by reversing the above work		
			(3)
(b)	**Ignore any imaginary solutions in (b)**		
	$4\left(\dfrac{e^x - e^{-x}}{2}\right) - 3\left(\dfrac{e^x + e^{-x}}{2}\right) = 3$	Substitutes the **correct** exponential forms for sinh*x* and cosh*x*	M1
	$e^x - 7e^{-x} = 6$		
	$e^{2x} - 6e^x - 7 = 0$	Obtains a quadratic in e^x	M1
	$(e^x + 1)(e^x - 7) = 0 \Rightarrow e^x = ...$	Attempt to solve their 3TQ in e^x as far as $e^x = ..$	M1
	$x = \ln 7$ or awrt 1.95		A1
			(4)
			Total 7

	Alternatives for (b)		
	$4\sinh x - 3\cosh x = 3 \Rightarrow 4\sinh x = 3 + 3\cosh x$ $\Rightarrow 7\cosh^2 x - 18\cosh x - 25 = 0$		M1
	M1: Attempt to square correctly and obtains a quadratic in sinh*x*		
	$7\cosh^2 x - 18\cosh x - 25 = 0 \Rightarrow (7\cosh x - 25)(\cosh x + 1) = 0$		
	$\cosh x = \frac{25}{7} \Rightarrow \frac{e^x + e^{-x}}{2} = \frac{25}{7} \Rightarrow 7e^{2x} - 50e^x + 7 = 0$		
	Uses the correct form of cosh*x* in terms of exponentials to obtain a 3TQ in e^x		M1
	$7e^{2x} - 50e^x + 7 = 0 \Rightarrow (7e^x - 1)(e^x - 7) = 0 \Rightarrow e^x = ...$		M1
	Attempt to solve their 3TQ as far as $e^x = ..$		
	$x = \ln 7$ or awrt 1.95	No other values	A1
	$4\sinh x - 3\cosh x = 3 \Rightarrow 4\sinh x - 3 = 3\cosh x$ $\Rightarrow 7\sinh^2 x - 24\sinh x = 0$		M1
	M1: Attempt to square correctly and obtains a quadratic in cosh*x*		
	$7\sinh^2 x - 24\sinh x = 0 \Rightarrow \sinh x(7\sinh x - 24) = 0$		
	$\sinh x = \frac{24}{7} \Rightarrow \frac{e^x - e^{-x}}{2} = \frac{24}{7} \Rightarrow 7e^{2x} - 48e^x - 7 = 0$		
	Uses the correct form of sinh*x* in terms of exponentials to obtain a 3TQ in e^x		M1
	$7e^{2x} - 48e^x - 7 = 0 \Rightarrow (7e^x + 1)(e^x - 7) = 0 \Rightarrow e^x = ...$		M1
	Attempt to solve their 3TQ as far as $e^x = ..$		
	$x = \ln 7$ or awrt 1.95	No other values	A1
	$4\sinh x - 3\cosh x = 3 \Rightarrow 4\tanh x - 3 = 3\operatorname{sech} x$ $\Rightarrow 25\tanh^2 x - 24\tanh x = 0$		M1
	M1: Attempt to square correctly and obtains a quadratic in tanh*x*		
	$25\tanh^2 x - 24\tanh x = 0 \Rightarrow \tanh x(25\tanh x - 24) = 0$		
	$\tanh x = \frac{24}{25} \Rightarrow \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{24}{25} \Rightarrow e^{2x} = 49$		
	Uses the correct form of tanh*x* in terms of exponentials to obtain a 2TQ in e^x		M1
	$e^{2x} = 49 \Rightarrow e^x = ...$		M1
	Attempt to solve their 2TQ as far as $e^x = ..$		
	$x = \ln 7$ or awrt 1.95	No other values	A1

Question	Scheme		Marks
5.	$\frac{dy}{dx} = \left(\frac{1}{1-\frac{x^2}{1+x^2}}\right) . \left(\frac{(1+x^2)^{\frac{1}{2}} - x^2(1+x^2)^{-\frac{1}{2}}}{(1+x^2)}\right)$ NB $\frac{(1+x^2)^{\frac{1}{2}} - x^2(1+x^2)^{-\frac{1}{2}}}{(1+x^2)} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$	M1: Correct form for the derivative of artanh$x$ using $\frac{x}{\sqrt{1+x^2}}$. M1: Correct quotient or product rule on $\frac{x}{\sqrt{1+x^2}}$ A1: Completely correct expression	M1M A1
	` $= \frac{1}{\sqrt{1+x^2}}$ **ag**	Correct solution with **no errors** seen	A1
			(4)
			Total 4
	Alternative 1		
	$y = \text{artanh}\frac{x}{\sqrt{1+x^2}} \Rightarrow \tanh y = \frac{x}{\sqrt{1+x^2}}$		
	$\text{sech}^2 y \frac{dy}{dx} = \left(\frac{(1+x^2)^{\frac{1}{2}} - x^2(1+x^2)^{-\frac{1}{2}}}{(1+x^2)}\right)$		
	$\frac{dy}{dx} = \left(\frac{1}{1-\frac{x^2}{1+x^2}}\right) . \left(\frac{(1+x^2)^{\frac{1}{2}} - x^2(1+x^2)^{-\frac{1}{2}}}{(1+x^2)}\right)$	M1: Divides by the correct form of sech^2y or their simplified sech^2y in terms of x M1: Correct quotient or product rule A1: Completely correct expression	M1M A1
	Then as above		
	Alternative 2		
	$y = \text{artanh}\frac{x}{\sqrt{1+x^2}} \Rightarrow \tanh y = \frac{x}{\sqrt{1+x^2}} \Rightarrow \tanh^2 y = \frac{x^2}{1+x^2}$		
	$2\tanh y\,\text{sech}^2 y \frac{dy}{dx} = \left(\frac{2x(1+x^2) - 2x^3}{(1+x^2)^2}\right)$		
	$\frac{dy}{dx} = \left(\frac{1}{1-\frac{x^2}{1+x^2}}\right) \times \frac{(1+x^2)^{\frac{1}{2}}}{2x} \times \frac{2x(1+x^2) - 2x^3}{(1+x^2)^2}$		M1M A1
	M1: Divides by the correct form of sech^2y and tanh$y$ in terms of x M1: Correct quotient or product rule A1: Completely correct expression		
	Then as above		

Question	Scheme		Marks
	In this question condone the use of *a* and/or *b* for *α* and *β*		
6(a)	Gradient $m=\dfrac{2\sin\beta-2\sin\alpha}{3\cos\beta-3\cos\alpha}$	Correct attempt at chord gradient – do not allow slips unless a correct method is clear	M1
	$y-2\sin\alpha=m(x-3\cos\alpha)$ or $y-2\sin\beta=m(x-3\cos\beta)$ or $y=mx+c$ and attempts to find c using P or Q	A correct straight line method using their **chord** gradient and the point P or the point Q	M1
	$y-2\sin\alpha=\dfrac{2\sin\beta-2\sin\alpha}{3\cos\beta-3\cos\alpha}(x-3\cos\alpha)$ $y-2\sin\beta=\dfrac{2\sin\beta-2\sin\alpha}{3\cos\beta-3\cos\alpha}(x-3\cos\beta)$ $y-2\sin\alpha=\dfrac{4\cos\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}{-6\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}(x-3\cos\alpha)$ $y=\dfrac{2\sin\beta-2\sin\alpha}{3\cos\beta-3\cos\alpha}x+2\sin\alpha-\dfrac{3\cos\alpha\left(2\sin\beta-2\sin\alpha\right)}{3\cos\beta-3\cos\alpha}$ $y=-\dfrac{2\cos\frac{\alpha+\beta}{2}}{3\sin\frac{\alpha+\beta}{2}}x+2\sin\alpha+\dfrac{2\cos\alpha\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}}$		A1
	A correct equation for the chord **in any form**.		
	$3y\sin\frac{1}{2}(\alpha+\beta)+2x\cos\frac{1}{2}(\alpha+\beta)=6(\cos\alpha\cos\frac{1}{2}(\alpha+\beta)+\sin\alpha\sin\frac{1}{2}(\alpha+\beta))$ or $3y\sin\frac{1}{2}(\alpha+\beta)+2x\cos\frac{1}{2}(\alpha+\beta)=6(\cos\beta\cos\frac{1}{2}(\alpha+\beta)+\sin\beta\sin\frac{1}{2}(\alpha+\beta))$		
	$\frac{x}{3}\cos\frac{1}{2}(\alpha+\beta)+\frac{y}{2}\sin\frac{1}{2}(\alpha+\beta)=\cos\frac{1}{2}(\alpha-\beta)$ **ag**		A1cso
	This is cso – there must no errors in applying the factor formulae and sufficient working must be shown to justify the printed answer but allow $\cos\beta\cos\frac{1}{2}(\alpha+\beta)+\sin\beta\sin\frac{1}{2}(\alpha+\beta)=\cos\frac{\alpha-\beta}{2}$		
			(4)
(b)	$\left(\dfrac{3\cos\alpha+3\cos\beta}{2},\dfrac{2\sin\alpha+2\sin\beta}{2}\right)$ or $(3\cos\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha),2\sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha))$ or $(3\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta),2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta))$		B1
	Correct coordinates of mid-point in any form Coordinates must be in this order but condone **outer** brackets missing		
			(1)

Question	Scheme		Marks
(c)	Centre of chord is $(3\cos\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha), 2\sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha))$ Attempt factor formulae on both coordinates of mid-point at any stage in (c) May be implied by their $\pm\frac{y}{x}$ below		M1
	$\pm\frac{y}{x}=\pm\frac{2\sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha)}{3\cos\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha)}\left(=\pm\frac{2\sin\frac{1}{2}(\beta+\alpha)}{3\cos\frac{1}{2}(\beta+\alpha)}\right)$ Or $\pm\frac{x}{y}=\pm\frac{3\cos\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha)}{2\sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha)}\left(=\pm\frac{3\cos\frac{1}{2}(\beta+\alpha)}{2\sin\frac{1}{2}(\beta+\alpha)}\right)$		**d**M1A1
	M1: Obtains an expression for k or $-k$ or $\frac{1}{k}$ or $-\frac{1}{k}$ **Dependent on the previous M1 (factor formulae must have been used)** A1: Correct expression in any form		
	$m=-\frac{2\cos\frac{1}{2}(\beta+\alpha)}{3\sin\frac{1}{2}(\beta+\alpha)}$	Must be seen or used in (c)	B1
	$\frac{\sin\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta+\alpha)}=-\frac{2}{3m}$ So $\frac{y}{x}=\frac{2}{3}\left(-\frac{2}{3m}\right)\Rightarrow k=\frac{4}{9m}$		A1**cso**
			(5)
			Total 10

Question	Scheme		Marks
7.(a)	$y=\sqrt{r^2-x^2} \Rightarrow \frac{dy}{dx}=-x(r^2-x^2)^{-\frac{1}{2}}$ Or $2x+2y\frac{dy}{dx}=0$	$\frac{dy}{dx}=px(r^2-x^2)^{-\frac{1}{2}}$ Or $px+qy\frac{dy}{dx}=0$	M1
	Attempts to differentiate explicitly or implicitly to give one of the given forms		
	$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{r^2-x^2}$ or $1+\frac{x^2}{y^2}$	Substitutes their derivative into $1+\left(\frac{dy}{dx}\right)^2$	M1
	$=\frac{r^2-x^2+x^2}{r^2-x^2}=\frac{r^2}{r^2-x^2}$ *	cso	A1*
	This is cso and so there must be no errors e.g. $\frac{dy}{dx}=\frac{x}{y}$ could give the correct answer but loses the A1 but allow to show equivalence of lhs and rhs		
			(3)
(b)	$S=(2\pi)\int y\sqrt{\frac{r^2}{r^2-x^2}}\,dx$	M1: Use of $\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ using their answer to part (a) **(must be *y* and not y^2)** 2π **not** required here	M1A1
		A1: Correct expression including 2π (may be implied by later work but must appear before any integration)	
	$=(2\pi)\int\sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}\,dx$	Substitutes for *y* in terms of *x*. **Dependent on first M.**	**d**M1
	$=[2\pi rx]_{-r}^{r}$ or $[2\pi rx]_0^r$	Substitutes the limits *r* and –*r* or 0 and *r* into an expression of the form $k\pi rx$ and subtracts. The use of the 0 limit can be taken on trust if omitted. **Dependent on both previous method marks.**	**dd**M1
	If they reach $2\pi r^2$ correctly then double, then some justification is needed e.g. some mention of symmetry		
	$=4\pi r^2$ *	cso	A1
			(5)
	Note that $S=2\times 2\pi\int_0^r y\sqrt{\frac{r^2}{r^2-x^2}}\,dx$ followed by correct work could score full marks as could the correct use of $S=(2\pi)\int y\sqrt{\frac{r^2}{y^2}}dx$		
(c)	arc length $=\frac{\pi}{2}$	Ignore any working	B1
			(1)
			Total 9

Question	Scheme		Marks
8.(a)	$\mathbf{OA}=\begin{pmatrix}1\\-1\\0\end{pmatrix}, \mathbf{OB}=\begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{OC}=\mathbf{AB}=\begin{pmatrix}0\\2\\1\end{pmatrix}, \mathbf{BC}=\begin{pmatrix}-1\\1\\0\end{pmatrix}, \mathbf{AC}=\begin{pmatrix}-1\\3\\1\end{pmatrix}$		
	$\mathbf{AB}\times\mathbf{AC}=-\mathbf{i}-\mathbf{j}+2\mathbf{k}$ Or e.g. $\mathbf{BA}\times\mathbf{BC}=\mathbf{i}+\mathbf{j}-2\mathbf{k}$	M1: Attempt vector product for two sides of the triangle. If the method is unclear, at least 2 components must be correct. A1: Correct vector	M1A1
	Area ABC = $\frac{1}{2}\sqrt{1^2+1^2+2^2}$	Attempts $\frac{1}{2}\lvert\text{their}\mathbf{AB}\times\mathbf{AC}\rvert$ **Dependent on the first M**	**d**M1
	$\frac{1}{2}\sqrt{6}$	Accept equivalents or awrt 1.22	A1
	Note that triangles OAB and OBC have the same area but score 0/4 It must be triangle ABC		
			(4)
(b)	$\mathbf{b}\times\mathbf{c}=(\mathbf{i}+\mathbf{j}+\mathbf{k})\times(2\mathbf{j}+\mathbf{k})=-\mathbf{i}-\mathbf{j}+2\mathbf{k}$	Attempt $\mathbf{b}\times\mathbf{c}$. If the method is unclear, at least 2 components must be correct.	M1
	$=\left(\frac{1}{6}\right)(\mathbf{i}-\mathbf{j}).(-\mathbf{i}-\mathbf{j}+2\mathbf{k})=\left(\frac{1}{6}\right)(-1+1)=0$	M1: Attempt scalar product of **a** with their $\mathbf{b}\times\mathbf{c}$ to obtain a number not a vector. A1: Obtains = 0 with no errors (allow omission of $\frac{1}{6}$ for all 3 marks) Just $=\mathbf{a}.(-\mathbf{i}-\mathbf{j}+2\mathbf{k})=0$ would lose the A1	M1A1
			(3)
	Alternative		
	$(\mathbf{a}.\mathbf{b}\times\mathbf{c}=)\begin{vmatrix}1&-1&0\\1&1&1\\0&2&1\end{vmatrix}$	Writes this statement (allow other brackets provided the determinant is implied later)	M1
	$=(1-2)+1(1)-0=0$	M1: Clear attempt at determinant A1: Obtains = 0 with no errors (allow omission of $\frac{1}{6}$ for all 3 marks)	M1A1
(c)	Volume of tetrahedron (OABC) = 0 $\mathbf{a}=\mathbf{b}-\mathbf{c}$ oe or $\mathbf{c}=\mathbf{b}-\mathbf{a}$ oe **b** x **c** is perpendicular to **a** or **a** is parallel to **CB** All vectors/points lie in the same plane OABC is a parallelogram **a**, **b** and **c** are linearly dependent **Do not isw – if there are contradictory or wrong statements award B0**		B1
			(1)
			Total 8

Question	Scheme		Marks
9(a)	$\int (x^2+1)^{-n}\,dx = x(x^2+1)^{-n} + \int xn(x^2+1)^{-n-1}2x\,dx$		M1A1
	M1: Integration by parts in the correct direction A1: Correct expression (If the parts formula is not quoted and the expression is wrong, score M0A0)		
	$= x(x^2+1)^{-n} + 2n\int x^2(x^2+1)^{-n-1}dx$		
	$= x(x^2+1)^{-n} + 2n\int (x^2+1)^{-n} - (x^2+1)^{-n-1}dx$	Use of $x^2 = x^2+1-1$ or equivalent. **Dependent on the previous method mark.**	dM1
	$I_n = x(x^2+1)^{-n} + 2nI_n - 2nI_{n+1}$	Correctly replaces $\int (x^2+1)^{-n}\,dx$ and $\int (x^2+1)^{-n-1}dx$ by I_n and I_{n+1}. **Dependent on both previous method marks.**	ddM1
	$I_{n+1} = \dfrac{x(x^2+1)^{-n}}{2n} + \dfrac{2n-1}{2n}I_n$	Correct completion to the printed answer with **no errors**.	A1cso
			(5)
(b)	$I_2 = \dfrac{x(x^2+1)^{-1}}{2} + \dfrac{1}{2}I_1$	Correct application of the given reduction formula **using *n* = 1 only**	M1
	$I_1 = \int \dfrac{dx}{x^2+1} = \arctan x(+C)$	$I_1 = k\arctan x$ (must be x and not just for arctan)	M1
	$I_2 = \dfrac{x}{2(x^2+1)} + \dfrac{1}{2}\arctan x(+C)$	Cao (constant not needed)	A1
			(3)
			Total 8

Extra Notes

2. (a) $\mathbf{M}^{-1} = \frac{1}{5}\begin{pmatrix} 5 & -10 & 8 \\ 0 & 0 & 1 \\ 0 & 5 & -4 \end{pmatrix}$

$\mathbf{M}^{\mathrm{T}}\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 41 & 4 \\ 2 & 4 & 5 \end{pmatrix}$

3. (b) Parts once then exponentials

$$I = \left[\tfrac{1}{2}\mathrm{e}^{2x}\sinh x\right]_0^1 - \int_0^1 \tfrac{1}{2}\mathrm{e}^{2x}\cosh x\,\mathrm{d}x = \left[\tfrac{1}{2}\mathrm{e}^{2x}\,\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}\right]_0^1 - \int_0^1 \tfrac{1}{2}\mathrm{e}^{2x}\,\frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}\,\mathrm{d}x$$

M1 integrates by parts and writes cosh$x$ as exponentials
A1 Correct expression

$$= \left[\tfrac{1}{2}\mathrm{e}^{2x}\,\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}\right]_0^1 - \left[\frac{1}{12}\mathrm{e}^{2x} + \frac{1}{4}\mathrm{e}^{x}\right]_0^1 = \left[\tfrac{1}{2}\left(\tfrac{1}{3}\mathrm{e}^{3x} - \mathrm{e}^{x}\right)\right]_0^1 == \tfrac{1}{2}\left(\tfrac{1}{3}\mathrm{e}^{3} - \mathrm{e}^{1}\right) - \tfrac{1}{2}\left(\tfrac{1}{3}\mathrm{e}^{0} - \mathrm{e}^{0}\right)$$

M1 $\int \mathrm{e}^{px}\,dx = q\mathrm{e}^{px}$ at least once and correct use of the limits 0 and 1

$$= \left(\frac{\mathrm{e}^3}{6} - \frac{\mathrm{e}}{2} + \frac{1}{3}\right) \text{ A1}$$

Any exact equivalent (allow e^1) but all like terms collected but isw following a correct answer.